TENTAMEN IMAGE PROCESSING

13-7-2009

A FORMULA SHEET IS INCLUDED ON PAGES 3-4

Put your name on all pages which you hand in, and number them. Write the total number of pages you hand in on the first page. Write clearly and not with pencil or red pen. You can answer in English or Dutch. ***Always motivate your answers.*** *You get 10 points for free. Success!*

Problem 1 (30 pt)
Consider the morphological skeleton of a binary image X by a structuring element B. Assume that the structuring element contains the origin: $\mathbf{0} = (0,0) \in B$.

a. Prove that $SK(X) = \emptyset$ if $B = \{\mathbf{0}\}$

b. Consider the case that the input image X is identical to the structuring element B. Prove that

$$S_n(B) = \begin{cases} \emptyset & \text{if } n = 0 \\ B & \text{if } n = 1 \\ \emptyset & \text{if } n \geq 2 \end{cases}$$

Give a geometrical interpretation of this result.

c. Define a *skeleton function* $skf(X)$ which can be used as a compact way to encode all skeleton sets $S_n(X), n = 0, 1, \ldots, N$.

d. The input image X can be exactly reconstructed from its skeleton sets $S_n(X)$ by:

$$X = \bigcup_{n=0}^{N} S_n(X) \underset{n}{\oplus} B$$

Describe (in a diagram or by using pseudocde) a recursive implementation implementation of this formula.

e. Describe a practical application of morphological skeletonization.

Problem 2 (30 pt)
Linear shift-invariant filtering of an image $f(x, y)$ in the frequency domain is defined as a multiplication:

$$G(u, v) = H(u, v)\, F(u, v). \qquad (1)$$

Here $F(u, v)$ and $G(u, v)$ are the 2-D Fourier transforms of the input and output image, respectively, and $H(u, v)$ is the filter transfer function. An example filter is the ideal highpass filter (IHPF), defined by

$$H(u, v) = \begin{cases} 0 & \text{if } D(u, v) \leq D_0 \\ 1 & \text{if } D(u, v) > D_0 \end{cases}$$

Here $D(u, v)$ is the distance of the point (u, v) to the origin of the frequency domain, and D_0 is the cut-off radius; see Figure 1.

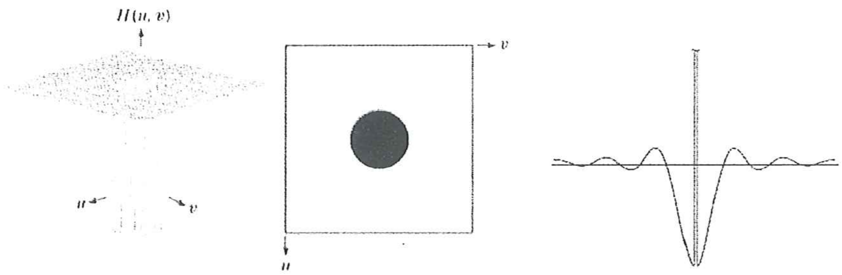

FIGURE 1: *Left and middle: perspective plot and image representation of a 2-D IHPF. Right: representation of a 1-D IHPF in the spatial domain.*

a. What is the purpose of highpass filtering?

b. Give the general formula corresponding to equation (1) which represents the filtering operation in the *spatial domain.*

c. Consider a one-dimensional IHPF, that is, $H(u) = 0$ for $-D_0 \leq u \leq D_0$ and $H(u) = 1$ elsewhere. Show that the corresponding spatial representation $h(x)$ is given by

$$h(x) = \delta(x) - 2D_0 \operatorname{sinc}(2D_0 x) \tag{2}$$

A graph of $h(x)$ is shown in Figure 1(c).

d. A typical artefact caused by IHPF is ringing. Explain this effect by considering the 1-D spatial representation of this filter, as given by equation (2).

e. How do the ringing artefacts change when the cut-off radius D_0 is increased?

Problem 3 (30 pt)
In this problem we consider image segmentation.

a. The main approaches to segmentation can be divided into *edge-based* and *region-based.* Explain the difference between these two approaches. Give one example of each approach, with a clarification of the basic computations involved.

b. Another subdivision of segmentation algorithms is into *global* and *local* ones. Again explain the difference between these two approaches, and give one example of each.

c. Several segmentation algorithms require *threshold selection.* Explain the principles of Otsu's method for threshold selection, and discuss for which image types it will give better results then heuristic threshold selection.

(*Formula sheet on next page*)

Formula sheet

Co-occurrence matrix $g(i,j) = \{$no. of pixel pairs with grey levels (z_i, z_j) satisfying predicate $Q\}$, $1 \leq i,j \leq L$

Convolution, 2-D discrete $(f \star h)(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)\, h(x-m, y-n)$, for $x = 0,1,2,\ldots,M-1$, $y = 0,1,2,\ldots,N-1$

Convolution Theorem, 2-D discrete $\mathcal{F}\{f \star h\}(u,v) = F(u,v)\, H(u,v)$

Distance measures Euclidean: $D_e(p,q) = \sqrt{(p_1-q_1)^2 + (p_2-q_2)^2}$, City-block: $D_4(p,q) = |p_1-q_1| + |p_2-q_2|$, Chessboard: $D_8(p,q) = \max(|p_1-q_1|, |p_2-q_2|)$

Entropy, source $H = -\sum_{j=1}^{J} P(a_j) \log P(a_j)$

Entropy, estimated for L-level image: $\tilde{H} = -\sum_{k=0}^{L-1} p_r(r_k) \log_2 p_r(r_k)$

Error, root-mean square $e_{\text{rms}} = \left[\frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left(\widehat{f}(x,y) - f(x,y)\right)^2\right]^{\frac{1}{2}}$

Exponentials $e^{ix} = \cos x + i \sin x$; $\cos x = (e^{ix} + e^{-ix})/2$; $\sin x = (e^{ix} - e^{-ix})/2i$

Filter, inverse $\widehat{\mathbf{f}} = \mathbf{f} + \mathbf{H}^{-1}\mathbf{n}$, $\widehat{F}(u,v) = F(u,v) + \frac{N(u,v)}{H(u,v)}$

Filter, parametric Wiener $\widehat{\mathbf{f}} = \left(\mathbf{H}^t\mathbf{H} + K\,\mathbf{I}\right)^{-1} \mathbf{H}^t\mathbf{g}$, $\widehat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + K}\right] G(u,v)$

Fourier series of signal with period T: $f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{i\frac{2\pi n}{T}t}$, with Fourier coefficients: $c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-i\frac{2\pi n}{T}t}\, dt$, $\quad n = 0, \pm1, \pm2, \ldots$

Fourier transform 1-D (continuous) $F(\mu) = \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\mu t}\, dt$

Fourier transform 1-D, inverse (continuous) $f(t) = \int_{-\infty}^{\infty} F(\mu)\, e^{i2\pi\mu t}\, d\mu$

Fourier Transform, 2-D Discrete $F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-i2\pi(u\,x/M + v\,y/N)}$ for $u = 0,1,2,\ldots,M-1$, $v = 0,1,2,\ldots,N-1$

Fourier Transform, 2-D Inverse Discrete $f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v)\, e^{i2\pi(u\,x/M + v\,y/N)}$ for $x = 0,1,2,\ldots,M-1$, $y = 0,1,\ldots,N-1$

Fourier spectrum Fourier transform of $f(x,y)$: $F(u,v) = R(u,v) + i\,I(u,v)$, Fourier spectrum: $|F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)}$, phase angle: $\phi(u,v) = \arctan\left(\frac{I(u,v)}{R(u,v)}\right)$

Gaussian function mean μ, variance σ^2: $G_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2}$

Gradient $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

Histogram $h(m) = \#\{(x,y) \in D : f(x,y) = m\}$. Cumulative histogram: $\mathsf{P}(\ell) = \sum_{m=0}^{\ell} h(m)$

Impulse, discrete $\delta(0) = 1, \delta(x) = 0$ for $x \in \mathbb{N} \backslash \{0\}$

Impulse, continuous $\delta(\infty) = 1, \delta(x) = 0$ for $x \neq 0$, with $\int_{-\infty}^{\infty} f(t)\, \delta(t - t_0)\, dt = f(t_0)$

Impulse train $s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$, with Fourier transform $S(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\mu - \frac{n}{\Delta T})$

Laplacian $\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

Laplacian-of-Gaussian $\nabla^2 G_\sigma(x,y) = -\frac{2}{\pi\sigma^4}\left(1 - \frac{r^2}{2\sigma^2}\right) e^{-r^2/2\sigma^2} \quad (r^2 = x^2 + y^2)$

Morphology

Dilation $\delta_A(X) = X \oplus A = \bigcup_{a\in A} X_a = \bigcup_{x\in X} A_x = \{h \in E : \check{A}_h \cap X \neq \emptyset\}$,
where $X_h = \{x + h : x \in X\}$, $h \in E$ and $\check{A} = \{-a : a \in A\}$

Erosion $\varepsilon_A(X) = X \ominus A = \bigcap_{a\in A} X_{-a} = \{h \in E : A_h \subseteq X\}$

Opening $\gamma_A(X) = X \circ A := (X \ominus A) \oplus A = \delta_A\varepsilon_A(X)$

Closing $\phi_A(X) = X \bullet A := (X \oplus A) \ominus A = \varepsilon_A\delta_A(X)$

Hit-or-miss transform $X \circledast (B_1, B_2) = (X \ominus B_1) \cap (X^c \ominus B_2)$

Thinning $X \otimes B = X \backslash (X \circledast B)$, **Thickening** $X \odot B = X \cup (X \circledast B)$

Morphological reconstruction Marker F, mask G, structuring element B:
$X_0 = F$, $X_k = (X_{k-1} \oplus B) \cap G, \quad k = 1, 2, 3, \ldots$

Morphological skeleton Image X, structuring element B: $SK(X) = \bigcup_{n=0}^{N} S_n(X)$,
$S_n(X) = X \underset{n}{\ominus} B \backslash (X \underset{n}{\ominus} B) \circ B$, where $X \underset{0}{\ominus} B = X$ and N is the largest integer such that $S_N(X) \neq \emptyset$

Grey value dilation $(f \oplus b)(x, y) = \max\limits_{(s,t)\in B} [f(x - s, y - t) + b(s, t)]$

Grey value erosion $(f \ominus b)(x, y) = \min\limits_{(s,t)\in B} [f(x + s, y + t) - b(s, t)]$

Grey value opening $f \circ b = (f \ominus b) \oplus b$

Grey value closing $f \bullet b = (f \oplus b) \ominus b$

Morphological gradient $g = (f \oplus b) - (f \ominus b)$

Top-hat filter $T_{\text{hat}} = f - (f \circ b)$, **Bottom-hat filter** $B_{\text{hat}} = (f \bullet b) - f$

Sampling of continuous function $f(t)$: $\tilde{f}(t) = f(t)\, s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\, \delta(t - n\Delta T)$.
Fourier transform of sampled function: $\tilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F(\mu - \frac{n}{\Delta T})$

Sampling theorem Signal $f(t)$, bandwidth $\mu_{\max}$: If $\frac{1}{\Delta T} \geq 2\mu_{\max}$, $f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta T)\ \text{sinc}\left[\frac{t - n\Delta T}{n\Delta T}\right]$.

Sampling: downsampling by a factor of 2: $\downarrow_2 (a_0, a_1, a_2, \ldots, a_{2N-1}) = (a_0, a_2, a_4, \ldots, a_{2N-2})$

Sampling: upsampling by a factor of 2: $\uparrow_2 (a_0, a_1, a_2, \ldots, a_{N-1}) = (a_0, 0, a_1, 0, a_2, 0, \ldots, a_{N-1}, 0)$

Set, circularity ratio $R_c = \frac{4\pi A}{P^2}$ of set with area A, perimeter P

Set, diameter $\text{Diam}(B) = \max\limits_{i,j} [D(p_i, p_j)]$ with p_i, p_j on the boundary B and D a distance measure

Sinc function $\text{sinc}\,(x) = \frac{\sin(\pi x)}{\pi x}$ when $x \neq 0$, and $\text{sinc}\,(0) = 1$. If $f(t) = A$ for $-W/2 \leq t \leq W/2$ and zero elswhere (block signal), then its Fourier transform is $F(\mu) = AW\text{sinc}\,(\mu W)$

Spatial moments of an $M \times N$ image $f(x, y)$: $m_{pq} = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} x^p\, y^q\, f(x, y), \quad p, q = 0, 1, 2, \ldots$

Statistical moments of distribution $p(i)$: $\mu_n = \sum_{i=0}^{L-1} (i - m)^n\, p(i)$, $m = \sum_{i=0}^{L-1} i\, p(i)$

Signal-to-noise ratio, mean-square $\text{SNR}_{\text{rms}} = \frac{\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \hat{f}(x,y)^2}{\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left(\hat{f}(x,y) - f(x,y)\right)^2}$

Wavelet decomposition with low pass filter h_ϕ, band pass filter h_ψ. For $j = 1, \ldots, J$:
Approximation: $c_j = \mathbf{H}c_{j-1} = \downarrow_2 (h_\phi * c_{j-1})$; Detail: $d_j = \mathbf{G}c_{j-1} = \downarrow_2 (h_\psi * c_{j-1})$

Wavelet reconstruction with low pass filter $\tilde{h}_\phi$, band pass filter $\tilde{h}_\psi$. For $j = J, J - 1, \ldots, 1$:
$c_{j-1} = \tilde{h}_\phi * (\uparrow_2 c_j) + \tilde{h}_\psi * (\uparrow_2 d_j)$

Wavelet, Haar basis $h_\phi = \frac{1}{\sqrt{2}}(1, 1)$, $h_\psi = \frac{1}{\sqrt{2}}(1, -1)$, $\tilde{h}_\phi = \frac{1}{\sqrt{2}}(1, 1)$, $\tilde{h}_\psi = \frac{1}{\sqrt{2}}(1, -1)$